

Generating Static Black Holes in Higher Dimensional Space-Times

Emanuel Gallo *

*Facultad de Matemática, Astronomía y Física
Universidad Nacional de Córdoba, Ciudad Universitaria
5000 Córdoba, Argentina*

Abstract

In this article we extend to higher dimensional space-times a recent theorem proved by Salgado [1] which characterizes a three-parameter family of static and spherically symmetric solutions to the Einstein Field Equations. As it happens in four dimensions, it is shown that the Schwarzschild, Reissner-Nordstrom and global monopole solutions in higher dimensions are particular cases from this family.

1 Introduction

Recently, Salgado [1] proved a simple theorem characterizing static spherically symmetric solutions to the Einstein's field equations in four dimensions when certain conditions on the energy-momentum tensor are imposed. This theorem allows us to find exact solutions like black-holes with different matter fields.

The solutions depend on three parameters (one of these being the cosmological constant). It can be easily shown that the Schwarzschild-de Sitter/Anti-de Sitter (SdS/SAdS) or Reissner-Nordstrom (RN) black-holes are particular cases from this family.

These results were independently obtained and used by Kiselev [2, 3] in the study of quintessence fields in black holes and dark matter.

Similar results were obtained by Giambó [4] in his study of anisotropic generalization of de Sitter space-time and by Dymnikova [5] in the study of a cosmological term as a source of mass.

On the other hand, the study of solutions of Einstein's equations in higher

*egallo@fis.uncor.edu

dimensional space-times has been very intense for almost a decade due to the request of extra dimensions for many physics theories (String theory, M theory, Brane-Worlds).

In particular, Myers and Perry [6] have found solutions to Einstein's equations representing black holes like Schwarzschild, Reissner-Nordstrom and Kerr in D-dimensions. Dianyan [7] has extended this work at SdS/SAdS and RN-dS, and Liu & Sabra [8] have studied general charged configurations in D-dimensional (A)dS spaces with relevant results to this work.

Other examples from solutions to the Einstein's equations in higher dimensions are the analysis of spherically symmetric perfect fluids [9], or the collapse of different fluids [10, 11, 12] or the study of wormholes [13, 14].

It is therefore interesting to characterize solutions to Einstein field equations in higher dimensions than four.

In this paper, we extend the theorem proved by Salgado to D-dimensional space-times, where SdS/SAdS and RN-dS are particular cases.

In section 2 we briefly review the RN-(dS/AdS) black holes in higher dimensions (for a more detailed study see [7]).

In section 3 we formulate the extension of the Salgado theorem to D-dimensions and finally in section 4, we will characterize the RN-(dS/AdS) and global monopole solutions in this family.

2 Reissner-Nordstrom (dS/AdS) Black Holes in Higher Dimensions

We begin by writing the Einstein-Maxwell equations in D dimensions with a cosmological constant Λ .

From the Einstein-Maxwell action in D dimensions

$$S = \int d^D x \sqrt{|g|} \left\{ R - 2\Lambda + \frac{\kappa}{8\pi} F_{ab} F^{ab} \right\}, \quad (1)$$

where

$$\kappa = \frac{8\pi G}{c^4}, \quad (2)$$

$$F_{ab} = A_{a;b} - A_{b;a}, \quad (3)$$

we obtain the following Einstein-Maxwell equations

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = \frac{\kappa}{4\pi} \left\{ F_a^c F_{bc} - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right\}, \quad (4)$$

$$F_{a;c}^c = 0, \quad (5)$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0. \quad (6)$$

Now, let us consider static and spherically symmetric solutions from these equations (We will use units where $G = c = 1$).

The most general static and spherically symmetric metric in D dimensions reads:

$$ds^2 = -N^2(r)dt^2 + A^2(r)dr^2 + r^2 d\Omega_{D-2}^2, \quad (7)$$

where

$$d\Omega_{D-2}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \dots + \prod_{n=1}^{D-3} \sin^2 \theta_n d\theta_{D-2}^2. \quad (8)$$

The only non trivial components from F_{ab} are:

$$F_{tr} = -F_{rt} = \frac{Q}{r^{D-2}}, \quad (9)$$

where Q represents an isolated point charge.

Solving the Einstein-Maxwell equations for the metric of eq.(7), we find ($D > 3$)

$$N^2(r) = \frac{1}{A^2(r)} = 1 - \frac{2M}{r^{D-3}} + \frac{2Q^2}{(D-3)(D-2)r^{2(D-3)}} - \frac{2\Lambda r^2}{(D-2)(D-1)} \quad (10)$$

with M a constant of integration.

This metric is a generalization to D dimension of Reissner-Nordstrom-dS/AdS, according to the sign of Λ .

For example, if we make $Q = 0$, $\Lambda > 0$, we obtain the Schwarzschild-dS black hole in D-dimensions. Its horizons were studied by Dianyan [7].

We will see in the next section (extending the Salgado's theorem) that these solutions are contained in a more general class of metrics.

3 Generating Static Black Holes

Now, following Salgado, we state and prove the following theorem:

Theorem 1 *Let (M, g_{ab}) be a D-dimensional space-time with $\text{sign}(g_{ab}) = D - 2$, $D \geq 3$, such that: (1) it is static and spherically symmetric, (2) it satisfies the Einstein field equations, (3) the energy-momentum tensor is given by $T_b^a = T_{[f]b}^a - \frac{\Lambda}{8\pi} \delta_b^a$, where $T_{[f]b}^a$ is the energy-momentum tensor of the matter fields, and Λ is a cosmological constant. (4) in the radial gauge coordinate system adapted to the symmetries of the space-time where $ds^2 = -N^2(r)dt^2 + A^2(r)dr^2 + r^2 d\Omega_{D-2}^2$, the energy-momentum tensor satisfies the conditions $T_{[f]t}^t = T_{[f]r}^r$ and $T_{[f]\theta_1}^{\theta_1} = \lambda T_{[f]r}^r$ ($\lambda = \text{const} \in \mathbb{R}$),*

(5) it possesses a regular Killing horizon or a regular origin. Then, the metric of the space-time is given by

$$ds^2 = - \left\{ 1 - \frac{2m(r)}{r^{D-3}} \right\} dt^2 + \left\{ 1 - \frac{2m(r)}{r^{D-3}} \right\}^{-1} dr^2 + r^2 d\Omega_{D-2}^2, \quad (11)$$

where

$$m(r) = \begin{cases} M + \frac{\Lambda r^{D-1}}{(D-2)(D-1)} & \text{if } C = 0 \\ M + \frac{\Lambda r^{D-1}}{(D-2)(D-1)} - \frac{8\pi C r^{(D-2)\lambda+1}}{(D-2)[(D-2)\lambda+1]} & \text{if } \lambda \neq -\frac{1}{D-2} ; C \neq 0 \\ M + \frac{\Lambda r^{D-1}}{(D-2)(D-1)} - \frac{8\pi C \ln(r)}{D-2} & \text{if } \lambda = -\frac{1}{D-2} ; C \neq 0 \end{cases} \quad (12)$$

$$T_{[f]b}^a = \frac{C}{r^{(D-2)(1-\lambda)}} \text{diag}[1, 1, \lambda, \dots, \lambda], \quad (13)$$

with M and C integration constants fixed by boundary conditions and fundamental constant of the underlying matter.

Proof. The proof follows exactly the same steps than Salgado's proof. From the hypothesis 1, we can write the metric in the following way

$$ds^2 = -N^2(r)dt^2 + A^2(r)dr^2 + r^2 d\Omega_{D-2}^2. \quad (14)$$

Introducing this metric into Einstein's equations (hypothesis 2)

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}, \quad (15)$$

we obtain the following equations for $A(r)$ and $N(r)$

$$\frac{\partial_r A}{Ar} + \frac{(D-3)(A^2-1)}{2r^2} = -\frac{8\pi A^2}{D-2} (T_{[f]t}^t - \frac{\Lambda}{8\pi}), \quad (16)$$

$$-\frac{\partial_r N}{Nr} + \frac{(D-3)(A^2-1)}{2r^2} = -\frac{8\pi A^2}{D-2} (T_{[f]r}^r - \frac{\Lambda}{8\pi}), \quad (17)$$

and from eq.(14) and eq.(15) we have that if $a \neq b$ then

$$T_b^a = 0, \quad (18)$$

and

$$T_{\theta_{D-2}}^{\theta_{D-2}} = \dots = T_{\theta_2}^{\theta_2} = T_{\theta_1}^{\theta_1}, \quad (19)$$

Rewriting $A(r)$ as

$$A(r) = \left\{ 1 - \frac{2m(r)}{r^{D-3}} \right\}^{-1/2}, \quad (20)$$

eq. (16) reads

$$\partial_r m = -\frac{8\pi r^{D-2}}{D-2}(T_{[f]t}^t - \frac{\Lambda}{8\pi}). \quad (21)$$

Moreover, subtracting eq. (16) from (17) we have

$$\frac{\partial_r(AN)}{AN} = -\frac{8\pi A^2 r}{D-2}(T_{[f]t}^t - T_{[f]r}^r), \quad (22)$$

and using the hypothesis 4, ($T_{[f]t}^t - T_{[f]r}^r = 0$), we find that $N = cA^{-1}$, where c is a constant which can be chosen equal to 1, if we redefine the time coordinate.

So

$$N(r) = A^{-1}(r) = \left\{1 - \frac{2m(r)}{r^{D-3}}\right\}^{1/2}, \quad (23)$$

with $m(r)$ given by eq. (21).

On the other hand, from the Einstein's equations, it follows that

$$\nabla_a T_b^a = 0, \quad (24)$$

and this equation in the metric eq.(14) reads

$$\partial_r T_{[f]r}^r = (T_{[f]t}^t - T_{[f]r}^r) \frac{\partial_r N}{N} - \frac{D-2}{r}(T_{[f]r}^r - T_{[f]\theta_1}^{\theta_1}). \quad (25)$$

Using the hypothesis 4 ($T_{[f]t}^t = T_{[f]r}^r$, $T_{[f]\theta_1}^{\theta_1} = \lambda T_{[f]r}^r$), one obtains

$$\partial_r T_{[f]r}^r = -\frac{D-2}{r} T_{[f]r}^r (1 - \lambda), \quad (26)$$

and by integration of this equation we have

$$T_{[f]r}^r = \frac{C}{r^{(D-2)(1-\lambda)}}. \quad (27)$$

Finally from eq. (21) we obtain $m(r)$

$$m(r) = \begin{cases} M + \frac{\Lambda r^{D-1}}{(D-2)(D-1)} & \text{if } C = 0 \\ M + \frac{\Lambda r^{D-1}}{(D-2)(D-1)} - \frac{8\pi C r^{(D-2)\lambda+1}}{(D-2)[(D-2)\lambda+1]} & \text{if } \lambda \neq -\frac{1}{D-2} ; C \neq 0 \\ M + \frac{\Lambda r^{D-1}}{(D-2)(D-1)} - \frac{8\pi C \ln(r)}{D-2} & \text{if } \lambda = -\frac{1}{D-2} ; C \neq 0 \end{cases} \quad (28)$$

◇.

As Salgado remarks, the hypothesis (5) is not used *a priori*, but it is indeed suggested by the condition $T_{[f]t}^t = T_{[f]r}^r$ (see [1]).

Note that if $D = 3$, and $\Lambda = C = 0$, the solution is a flat metric, because in three dimensions there is no curvature in vacuum.

We could do a study of the energy condition on the matter fields as in [1], but it gives the same results as in four dimensions, and it not will be repeated here.

The only change is on the *effective equation of state* as measured by an observer staying in rest in the coordinate system given in the theorem:

$$p_f = \frac{1}{D-1}(T_r^r + \sum_{i=1}^{D-2} T_{\theta_i}^{\theta_i}) = -\frac{[1 + (D-2)\lambda]}{D-1}\rho, \quad (29)$$

where p_f is the effective pressure and ρ is the energy-density of the matter defined by

$$\rho = -T_t^t = -T_r^r = -\frac{C}{r^{(D-2)(1-\lambda)}}. \quad (30)$$

For example, if we have a electromagnetic field, ($\lambda = -1$, see the next section) we get

$$p_f = \frac{D-3}{D-1}\rho, \quad (31)$$

and if $D = 4$ we have the well known state equation $p_f = \rho/3$.

4 Characterizing RN-(dS/AdS) and global monopoles solutions in higher dimensions

Now, we characterize some known solutions in this three-parameter family.

Let us begin with the Reissner-Nordstrom (dS/AdS) black holes.

For a static spherically symmetric electrical field we have

$$F_{tr} = -F_{rt} = \frac{Q}{r^{(D-2)}}, \quad (32)$$

and from the electromagnetic energy-momentum tensor

$$T_{ab} = \frac{1}{4\pi}(F_a^c F_{bc} - \frac{1}{4}g_{ab}F_{cd}F^{cd}), \quad (33)$$

we can see that

$$T_b^a = -\frac{Q^2}{8\pi r^{2(D-2)}}diag[1, 1, -1, \dots, -1]. \quad (34)$$

Then, we have

$$\lambda = -1, \quad (35)$$

and

$$C = -\frac{Q^2}{8\pi}. \quad (36)$$

Then, if $D > 3$ putting these values for the parameters into the metric from the theorem, we have all RN-(dS/AdS) metrics in D dimensions (eq.10). For example, if $D=4$, we can see that $A(r)$ and $N(r)$ reads

$$N(r) = A^{-1}(r) = \left\{ 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} \right\}^{1/2}, \quad (37)$$

which is the well-known Reissner-Nordstrom (dS/AdS) metric in four dimensions.

In the case $D=3$, (using the theorem) we see that the metric for a charged (dS/AdS) black hole in three dimensions reads

$$ds^2 = -\left\{ \widetilde{M} - \Lambda r^2 - 2Q^2 \ln r \right\} dt^2 + \left\{ \widetilde{M} - \Lambda r^2 - 2Q^2 \ln r \right\}^{-1} dr^2 + r^2 d\theta_1^2, \quad (38)$$

where $\widetilde{M} = 1 - 2M$.

Properties of these solutions and 2+1 black holes in general have been studied in [15, 16, 17].

Finally, we study another solution which represents a black hole with a trivial global monopole inside.

The energy-momentum tensor for a global monopole in D dimensions is given by

$$T_{ab} = \nabla_a \phi^i \nabla_b \phi_i - \frac{g_{ab}}{2} (\nabla \phi^i)^2 - g_{ab} \frac{\nu}{4} (\phi^i \phi_i - \eta^2)^2. \quad (39)$$

Then, if we have a trivial monopole [18]

$$\phi^i = \frac{\eta x^i}{r} \quad (i = 1, 2, \dots, D-1) \quad (40)$$

which remains in the vacuum state $V(\phi_i \phi^i) = \frac{\nu}{4} (\phi_i \phi^i - \eta^2)^2 = 0$ it can be shown that

$$T_b^a = -\frac{(D-2)\eta^2}{2r^2} \text{diag}[1, 1, \frac{D-4}{D-2}, \dots, \frac{D-4}{D-2}], \quad (41)$$

and then we note that this monopole satisfies the theorem's conditions on its energy-momentum tensor, with $\lambda = \frac{D-4}{D-2}$ and $C = -\frac{(D-2)\eta^2}{2}$.

If $D > 3$ we obtain a D-dimensional solution representing a global monopole inside a static and spherically symmetric black-hole:

$$N(r) = A^{-1}(r) = \left\{ 1 - \frac{2M}{r^{D-3}} - \frac{2\Lambda r^2}{(D-2)(D-1)} - \frac{8\pi\eta^2}{(D-3)} \right\}^{1/2} \quad (42)$$

For example, if $D = 4$, then $\lambda = 0$ and we have (with $\Lambda = 0$) the well-known solution

$$N(r) = A^{-1}(r) = \left\{ 1 - \frac{2M}{r} - 8\pi\eta^2 \right\}^{1/2}. \quad (43)$$

On the other hand, if $D = 3$ (i.e $\lambda = -1$) and $\Lambda = 0$, we can see (using the theorem) that the monopole's 3-metric is

$$N(r) = A^{-1}(r) = \left\{ \widetilde{M} - 8\pi\eta^2 \ln(r) \right\}^{1/2}, \quad (44)$$

with $\widetilde{M} = 1 - 2M$.

5 Conclusions

In this article, we have extended to D dimensions a simple theorem which characterizes spherically symmetric solutions to the Einstein field equations under certain conditions of the energy-momentum tensor.

It can be seen that there exist some matter fields with a parameter λ so that $T_{[f]\theta}^\theta = \lambda T_{[f]r}^r$, and this parameter characterizes (partially) the radial dependence of the metric generated by the matter fields.

For a static spherically symmetric electromagnetic field, it was shown that $\lambda = -1$ independently of the dimension.

On the other hand, for the case of global monopoles solutions, λ depends explicitly on the dimensionality, i.e., $\lambda = \frac{D-4}{D-2}$.

Finally, it is very interesting to ask if there exist another matter fields which satisfies the theorem's conditions in D dimensions.

6 Acknowledgements

The author thanks Carlos Kozameh for the reading and revision of this paper. The author is supported by CONICET.

References

- [1] M. Salgado. *A simple theorem to generate exact black-hole solutions.* Class.Quantum Grav. **20**, 4451 (2003).
- [2] V.V. Kiselev. *Quintessence and Black Holes.* Class.Quantum.Grav. **20** 1187, (2003)
- [3] V.V. Kiselev. *Quintessential solution of dark matter rotation curves and its simulation by extra dimensions.* gr-qc/0303031 (2003).

- [4] R. Giambó *Anisotropic generalization of de Sitter spacetime.* Class.Quant.Grav. **19** 4399, (2002)
- [5] I. Dymnikova *Cosmological term as a source of mass.* Class.Quantum.Grav. **19** 725, (2003).
- [6] R.C. Myers and M.J. Perry. *Black Holes in Higher Dimensional Space-times.* Ann. Phys. (N.Y.) **172**, 304(1986).
- [7] X. Dianyan. *Exact solutions of Einstein and Einstein-Maxwell equations in higher-dimensional spacetime.* Class.Quantum Grav. **5**, 871 (1988).
- [8] J. Liu and W. Sabra. *Charged configurations in (A)dS spaces.* hep-th/0307300 (2003).
- [9] K. D. Krori, P. Borgohain, and Kanika Das. *Spherically symmetric solutions in higher dimensions.* J. Math. Phys. **30**, 2315 (1989).
- [10] S.G. Ghosh and D.W. Deshkar. *Gravitational Collapse of Perfect Fluid in Self-Similar Higher Dimensional Space-Times.* Int.J.Mod.Phys. D **12** 913 (2003).
- [11] S.G. Ghosh and A. Banerjee. *Non-marginally bound inhomogeneous dust collapse in higher dimensional space-time.* Int.J.Mod.Phys. D **12** 639 (2003).
- [12] K. S.G. Ghosh and N. Dadhich. *Gravitational collapse of Type II fluid in higher dimensional space-times.* Phys.Rev. D **65** 127502 (2002).
- [13] K. A. DeBenedictis and A. Das. *Higher Dimensional Wormhole Geometries with Compact Dimensions.* Nucl.Phys. B **653** 279 (2003).
- [14] K. M. Cataldo, P. Salgado, and P. Minning. *Self-dual Lorentzian wormholes in n-dimensional Einstein gravity.* Phys.Rev. D **66** 124008 (2002).
- [15] D. Park and S. Yang. *Geodesics motions in 2+1 dimensional charged black holes.* Gen.Rel.Grav. **31** 1343 (1999).
- [16] I. Kogan. *About some exact solutions for 2+1 gravity coupled to gauge fields.* Mod.Phys.Lett. A **7** 2341 (1992).
- [17] M. Bañados, C. Teitelboim and J. Zanelli. *Geometry of the 2+1 black hole.* Phys.Rev. D **48** 1506 (1993).
- [18] A. Vilenkin and E. Shellard. *Cosmic Strings and Other Topological Defects.* Canbrige University Press (1994).